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THE RATE OF CONVERGENCE OF EULER APPROXIMATIONS FOR SOLUTIONS OF STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY FRACTIONAL BROWNIAN MOTION

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Abstract. The paper focuses on discrete-type approximations of solutions to non-homogeneous stochastic differential equations (SDEs) involving fractional Brownian motion (fBm). We prove that the rate of convergence for Euler approximations of solutions of pathwise SDEs driven by fBm with Hurst index $H > 1/2$ can be estimated by $O(\delta^{2H-1})$ (δ is the diameter of partition). For discrete-time approximations of Skorohod-type quasilinear equation driven by fBm we prove that the rate of convergence is $O(\delta^H)$.

1. Introduction

Many equations which arise in modeling of processes in physics, chemistry, biology, finance, contain randomness. This randomness is not always well modeled by the classical Gaussian white noise (Brownian motion) because of long-range dependence, or long memory, of the processes under consideration. In this case the appropriate model for the randomness is fractional Brownian motion. Recall that $B = (B_t)_{t\geq 0}$ is called fractional Brownian motion on a complete probability space $(\Omega, \mathcal{F}, P)$ with Hurst parameter $H \in (\frac{1}{2}, 1)$ if B is a centered Gaussian process with stationary increments and covariance $R_H(t,s) = E(B_t B_s) = \frac{1}{2}(t^{2H}+s^{2H}-|t-s|^{2H})$.

Numerical solution via time discretization of SDEs driven by Brownian motion has long history. We refer to the monograph [6], which contains almost complete theory of numerical solution of such SDEs with regular coefficients. The paper [7] is devoted to Euler approximations for SDEs driven by semimartingales. Concerning numerical solution of SDEs driven by fBm, we mention first the paper [3], where equations with modified fBm that represents a special semimartingale are studied (recall that fBm itself is not a semimartingale). Papers [10, 9] study Euler approximations for homogeneous one-dimensional SDEs with bounded coefficients having bounded derivatives up to third order, driven by fBm, and prove that error of approximation is a.s. equivalent to $\delta^{2H-1}\xi_t$, and the process ξ_t is given explicitly. These papers also discuss Crank–Nicholson and Milstein schemes for SDEs driven by fBm. The methods used by the authors cannot be applied to our case, because they require homogeneity and high regularity of the coefficients.

We consider the stochastic differential equation on R^d

$$(1)\quad X_t^i = X_0^i + \sum_{j=1}^m \int_0^t \sigma^{ij}(s, X_s)dB_s^j + \int_0^t b^i(s, X_s)ds, \quad i = 1, ..., d, \quad t \in [0, T]$$

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where the processes $B^i, i = 1, ..., m$ are fractional Brownian motions with Hurst parameter H, X_0 is a d-dimensional random variable, the coefficients $\sigma^{i\,j}, b^i : \Omega \times [0,T] \times R^d \to R$ are measurable functions.

The integral in the right-hand side of (1) can be understood in the pathwise sense defined in [12, 11] or in Wick–Skorohod sense [1]. We treat the pathwise case first. We remind that the pathwise integral w.r.t. a one-dimensional fBm B can be defined as

$$\int_a^b f dB = \int_a^b \big(D_{a+}^{\alpha} f\big)(s)\big(D_{b-}^{1-\alpha} B_{b-}\big)(s) ds,$$

where

$$\big(D_{a+}^{\alpha} f\big)(s) = \frac{1}{\Gamma(1-\alpha)}\left[\frac{f(s)}{(s-a)^{\alpha}} + \alpha \int_a^s \frac{f(s)-f(u)}{(s-u)^{\alpha+1}} du\right]\mathbf{1}_{(a,b)}(s)$$

and

$$\big(D_{b-}^{1-\alpha} B_{b-}\big)(s) = \frac{e^{-i\pi\alpha}}{\Gamma(\alpha)}\left[\frac{B_{b-}(s)}{(b-s)^{1-\alpha}} + (1-\alpha)\int_s^b \frac{B_{b-}(s)-B_{b-}(u)}{(u-s)^{2-\alpha}} du\right]\mathbf{1}_{(a,b)}(s)$$

are fractional derivatives of corresponding orders,

$$B_{b-}(s) = \big(B_s - B_b\big)\mathbf{1}_{(a,b)}(s).$$

The integral exists for any $\alpha \in (1-H, \nu)$ if, for example, $f \in C^{\nu}(a,b)$ with $\nu + H > 1$. Moreover, in this case pathwise integral admits an estimate

$$\left|\int_a^b f dB\right| \le C_0(\omega)\left[\int_a^b \frac{|f(s)|}{(s-a)^{\alpha}} ds + \int_a^b \int_a^s \frac{|f(s)-f(u)|}{(s-u)^{\alpha+1}} du\, ds\right], \tag{2}$$

where $C_0(\omega) = C \cdot \sup_{a<s<b} \left|D_{b-}^{1-\alpha} B_{b-}(s)\right| < \infty$ a.s.

Denote $\sigma = (\sigma^{ij})_{d\times m}, b = (b^i)_{d\times 1}$ and for a matrix $A = (a^{ij})_{d\times m}$, and a vector $y = (y^i)_{d\times 1}$ denote $|A| = \sum_{i,j} |a^{ij}|, |y| = \sum_i |y^i|$.

We suppose that the coefficients satisfy the following assumptions

(A) $\sigma(t,x)$ is differentiable in x and there exist such $M > 0, 1-H < \beta \le 1, \frac{1}{H} - 1 < \kappa \le 1$ and for any $N > 0$ there exists such $M_N > 0$ that
 1) $|\sigma(t,x) - \sigma(t,y)| \le M|x-y|, x, y \in R^d, t \in [0,T]$;
 2) $|\partial_{x_i}\sigma(t,x) - \partial_{x_i}\sigma(t,y)| \le M_N|x-y|^{\kappa}, |x|, |y| \le N,\ t \in [0,T]$;
 3) $|\sigma(t,x) - \sigma(s,x)| + |\partial_{x_i}\sigma(t,x) - \partial_{x_i}\sigma(s,x)| \le M|t-s|^{\beta}, x \in R^d, t, s \in [0,T]$.

(B) 1) for any $N > 0$ there exists $L_N > 0$ such that

$$|b(t,x) - b(t,y)| \le L_N|x-y|,\ \ |x|, |y| \le N,\ \ t \in [0,T];$$

 2) $|b(t,x)| \le L(1+|x|)$.

As it was stated in [11], under conditions (A)–(B) the equation (1) has the unique solution $\{X_t,\ t \in [0,T]\}$, and for a.a. $\omega \in \Omega$ this solution belongs to $C^{H-\rho}[0,T]$ for any $0 < \rho < H$. Now, let $t \in [0,T], \delta = \frac{T}{N}, \tau_n = \frac{nT}{N} = n\delta, n = 0, ..., N$. Consider discrete Euler approximations of solution of equation (1),

$$\widetilde{Y}^{i,\delta}_{\tau_{n+1}} = \widetilde{Y}^{i,\delta}_{\tau_n} + b^i(\tau_n, \widetilde{Y}^{\delta}_{\tau_n})\delta + \sum_{j=1}^{m} \sigma^{ij}(\tau_n, \widetilde{Y}^{\delta}_{\tau_n})\Delta B^j_{\tau_n}, \quad \widetilde{Y}^{i,\delta}_0 = X^i_0,$$

and corresponding continuous interpolations

$$(3)\ \ Y_t^{i,\delta} = \widetilde{Y}_{\tau_n}^{i,\delta} + b^i(\tau_n, \widetilde{Y}_{\tau_n}^{\delta})(t-\tau_n) + \sum_{j=1}^{m} \sigma^{ij}(\tau_n, \widetilde{Y}_{\tau_n}^{\delta})(B_t^j - B_{\tau_n}^j), \quad t \in [\tau_n, \tau_{n+1}].$$

Continuous interpolations satisfy the equation

$$(4)\qquad Y_t^{i,\delta} = X_0^i + \int_0^t b^i(t_u, Y_{t_u}^{\delta})du + \sum_{j=1}^{m} \int_0^t \sigma^{i\,j}(t_u, Y_{t_u}^{\delta})dB_u^j,$$

where $t_u = \tau_{n_u}$, $n_u = \max\{n : \tau_n \le u\}$.

For simplicity we denote the vector of solutions as $X_t = (X_t^i)_{i=1,\dots,d}$, vector of continuous approximations as $Y_t^{\delta} = (Y_t^{\delta,i})_{i=1,\dots,d}$. Throughout the paper, C denotes a generic constant, whose value is not important and may change from line to line, and we write $C(\cdot)$, if the dependence on some parameters is crucial.

The paper is organized as follows. Sections 2 and 3 are devoted to equations with pathwise integral. Section 2 describes the growth and Hölder properties of approximations Y_t^{δ}. We use here growth and Hölder estimates of solution of corresponding pathwise equations from the paper [11]. Section 3 contains estimates of rate of convergence for Euler approximations of the solutions of pathwise equations. It is well-known that in the case $H = \frac{1}{2}$, when we have SDE with Itô integral with respect to Wiener process, the rate of convergence is $O(\delta^{1/2})$. It is natural to expect that in our case the rate of convergence might be $O(\delta^H)$. Nevertheless, the rate of convergence is only of order $O(\delta^{2H-1})$ unless the diffusion coefficient is constant (where the rate is $O(\delta^H)$, as expected). Section 4 is devoted to discrete-time approximations of solutions of SDEs with stochastic divergence integral with respect to the fBm (otherwise known as Skorohod integral, or fractional white noise integral). It is shown that the rate of convergence is $O(\delta^H)$ in this case. The better rate of convergence is mainly because in the pathwise case there is no "Itô compensator" for the integral.

2. Some properties of Euler approximations for solutions of pathwise equations

In this section we consider growth and Hölder properties of the approximation process $\{Y_t^{\delta}, t \in [0,T]\}$. We need some additional notations. Denote $\varphi_{u,v} := \left|Y_{t_u}^{\delta} - Y_v^{\delta}\right|(u-v)^{-\alpha-1}$ for $0 < v < t_u < T$, $0 < \alpha < 1$, $X_t^* := \sup_{0\le s\le t}|X_s|$, $Y_t^{\delta,*} := \sup_{0\le s\le t}\left|Y_s^{\delta}\right|$. Further, for any $0 < \rho < H$ there exists such $C = C(\omega,\rho)$ that for any $0 < v < u$

$$(5)\qquad |B_u - B_v| \le C(\omega,\rho)(u-v)^{H-\rho}.$$

We shall use the following statement [11, Lemma 7.6]

Proposition 1. *Let $0 < \alpha < 1$, $a, b > 0$, $x : R_+ \to R_+$ be a continuous function such that for each t*

$$x_t \le a + bt^{\alpha}\int_0^t (t-s)^{-\alpha}s^{-\alpha}x_s ds.$$

Then $x_t \le ac_\alpha \exp\left\{d_\alpha t b^{1/(1-\alpha)}\right\}$, where $c_\alpha = 4e^2\frac{\Gamma(1-\alpha)}{1-\alpha}$, $d_\alpha = 2\big(\Gamma(1-\alpha)\big)^{1/(1-\alpha)}$, $\Gamma(\cdot)$ is Euler's Gamma function.

We also establish technical lemma, which will be used later.

Lemma 1. *There exists such $C = C_\alpha > 0$ that for any $s \in [0, T]$, $s \neq t_s$ and $\delta \leq 1$, $\alpha \in (0, 1)$ it holds*

$$J := \int_0^{t_s} (s-u)^{-\alpha-1} \int_u^{t_u} (v - t_v)^{-\alpha} dv\, du \leq C\delta^{-\alpha}.$$

Proof. Evidently,

$$J = \int_0^{t_s} (v-t_v)^{-\alpha} \int_0^{v} (s-u)^{-\alpha-1} du\, dv \leq \alpha^{-1} \int_0^{t_s} (v-t_v)^{-\alpha} (s-v)^{-\alpha} dv.$$

Let $t_s = n\delta$ for some $0 < n \leq N$. Then

$$\int_0^{t_s} (v-t_v)^{-\alpha}(s-v)^{-\alpha} dv = \sum_{k=0}^{n-2} \int_{\tau_k}^{\tau_{k+1}} + \int_{(n-1)\delta}^{(2n-1)\delta/2} + \int_{(2n-1)\delta/2}^{n\delta}.$$

We estimate the integrals individually:

$$\begin{aligned}
&\int_{\tau_k}^{\tau_{k+1}} \leq (s-\tau_{k+1})^{-\alpha} \int_{\tau_k}^{\tau_{k+1}} (v-t_v)^{-\alpha} dv \leq (1-\alpha)^{-1}(s-\tau_{k+1})^{-\alpha}\delta^{1-\alpha},\\
&\int_{(n-1)\delta}^{(2n-1)\delta/2} \leq (\delta/2)^{-\alpha} \int_{(n-1)\delta}^{(2n-1)\delta/2} (v-t_v)^{-\alpha} dv \leq C\delta^{1-2\alpha},\\
&\int_{(2n-1)\delta/2}^{n\delta} \leq (\delta/2)^{-\alpha} \int_{(2n-1)\delta/2}^{n\delta} (s-v)^{-\alpha} dv \leq C\delta^{1-2\alpha}.
\end{aligned}$$

Therefore

$$\begin{aligned}
J &\leq C\delta^{1-2\alpha} + \delta^{-\alpha} \sum_{k=0}^{n-2} (s-\tau_{k+1})^{-\alpha}\delta \leq C\delta^{1-2\alpha} + \delta^{-\alpha} \int_0^{n\delta} (s-v)^{-\alpha} dv\\
&\leq C\delta^{1-2\alpha} + C\delta^{-\alpha} \leq C\delta^{-\alpha}.
\end{aligned}$$

□

Theorem 1. *(i) Let the conditions* (A)–(B) *hold and*

(C) 1) $\quad |\sigma(t,x)| \le C(1+|x|)$.

Then for any $\varepsilon > 0$ and $0 < \rho < H$ there exists $\delta_0 > 0$ and $\Omega_{\varepsilon,\delta_0,\rho} \subset \Omega$ such that $P(\Omega_{\varepsilon,\delta_0,\rho}) > 1-\varepsilon$ and for any $\omega \in \Omega_{\varepsilon,\delta_0,\rho}$, $\delta < \delta_0$ one has $\left|Y_t^\delta\right| \le C(\omega)$, $\left|Y_{t_s}^\delta - Y_{t_r}^\delta\right| \le C(\omega)(t_s-t_r)^{H-\rho}$, $0 \le r < s \le T$.

(ii) If, instead of (A), 2) *and* (C) *we assume that b and σ are bounded functions, then $\left|Y_t^\delta\right| \le C(\omega)$, $\left|Y_s^\delta - Y_r^\delta\right| \le C(\omega)(s-r)^{H-\rho}$, $0 \le r < s \le T$.*

In both cases $C(\omega)$ does not depend on δ.

Proof. We can assume that $\delta \le 1$. It follows immediately from (A), 1) and 3) and (4) that for any $\alpha \in (1-H, \beta \wedge 1/2)$

(6)

$$\begin{aligned}
|Y_t^{i,\delta}| &\le \left|X_0^i\right| + \int_0^t \left|b^i(t_u, Y_{t_u}^\delta)\right| du + \sum_{j=1}^m \left|\int_0^t \sigma^{i\,j}(t_u, Y_{t_u}^\delta) dB_u^H\right| \\
&\le \left|X_0^i\right| + L\int_0^t \Big(1+\left|Y_{t_u}^\delta\right|\Big) du + C_0(\omega)\sum_{j=1}^m \int_0^t \left|\sigma^{i\,j}(t_u, Y_{t_u}^\delta)\right| u^{-\alpha} du \\
&\quad + C_0(\omega)\sum_{j=1}^m \int_0^t\int_0^r \left|\sigma^{i\,j}(t_r, Y_{t_r}^\delta) - \sigma^{i\,j}(t_u, Y_{t_u}^\delta)\right| (r-u)^{-\alpha-1} du\, dr \\
&\le \left|X_0^i\right| + \Big(C_0(\omega)\frac{T}{1-\alpha} + LT\Big) + \big(C_0(\omega) + CT^\alpha\big)\int_0^t \left|Y_{t_u}^\delta\right| u^{-\alpha} du \\
&\quad + MC_0(\omega)\int_0^t\int_0^{t_r} \Big((t_r-t_u)^\beta + \left|Y_{t_r}^\delta - Y_u^\delta\right| + \left|Y_u^\delta - Y_{t_u}^\delta\right|\Big)(r-u)^{-\alpha-1} du\, dr.
\end{aligned}$$

(We use here the equality $t_r = t_u$ for $t_r \le u < r$.) Denote $C_1(\omega) := m\big(C_0(\omega)\frac{T^{1-\alpha}}{1-\alpha} + LT\big) + |X_0|$, $C_2(\omega) := m(C_0(\omega) + CT^\alpha)$. Further, note that $t_r - t_u \le r - u + \delta$. Also, it follows from representations (3) that for any $\rho \in (0, H)$

$$\begin{aligned}
\left|Y_u^\delta - Y_{t_u}^\delta\right| &\le L\Big(1+\left|Y_{t_u}^\delta\right|\Big)(u-t_u) + C\cdot C(\omega,\rho)\Big(1+\left|Y_{t_u}^\delta\right|\Big)(u-t_u)^{H-\rho} \\
&\le C_3(\omega)\Big(1+\left|Y_{t_u}^\delta\right|\Big)(u-t_u)^{H-\rho},
\end{aligned} \tag{7}$$

where $C_3(\omega) = LT^{1-H-\rho} + C\cdot C(\omega,\rho)$.

Moreover, for $\beta > \alpha$

$$\begin{aligned}
P_t &:= \int_0^t\int_0^{t_r} (t_r-t_u)^\beta (r-u)^{-\alpha-1} du\, dr \le \int_0^t\int_0^{t_r} \big((r-u)^\beta + \delta^\beta\big)(r-u)^{-\alpha-1} du\, dr \\
&\le (\beta-\alpha)^{-1}\int_0^t r^{\beta-\alpha} dr + \alpha^{-1}\delta^\beta \int_0^t (r-t_r)^{-\alpha} dr,
\end{aligned}$$

and for any $k \ge 0$ and any power $\pi > -1$

$$\int_{\tau_k}^{\tau_{k+1}} (r-t_r)^\pi dr = \int_{\tau_k}^{\tau_{k+1}} (r-\tau_k)^\pi dr = C_1\delta^{\pi+1} \text{ with } C_1 = (\pi+1)^{-1},$$

whence

$$\int_0^t (r-t_r)^{-\alpha} dr \le \int_0^T (r-t_r)^{-\alpha} dr = C_1 N\delta^{1-\alpha} = C_1\delta^{-\alpha}. \tag{8}$$

Therefore

$$P_t \le C_1 T^{\beta-\alpha+1} + \alpha^{-1} C_1 \delta^{\beta-\alpha} \le C_1 T^{\beta-\alpha+1} + \alpha^{-1} C_1 =: C_2. \tag{9}$$

Estimate now

$$Q_t := \int_0^t \int_0^{t_r} \left|Y_u^\delta - Y_{t_u}^\delta\right| (r-u)^{-\alpha-1} du\, dr,$$

using (7) and (8):

$$\begin{aligned} Q_t &\le \left(1+Y_t^{\delta,*}\right) \int_0^t \int_0^{t_r} (u-t_u)^{H-\rho}(r-u)^{-\alpha-1} du\, dr \\ &\le C_3(\omega)\left(1+Y_t^{\delta,*}\right)\delta^{H-\rho}\alpha^{-1} \int_0^t (r-t_r)^{-\alpha} dr \le C_4(\omega)\left(1+Y_t^{\delta,*}\right)\delta^{H-\alpha-\rho}, \end{aligned} \tag{10}$$

with $C_4(\omega) = C_3(\omega)\alpha^{-1}\cdot C_1$. Note that $Y_t^{\delta,*} := \sup_{0\le s\le t}\left|Y_s^\delta\right| < \infty$ for any $t \in [0,T]$ a.s. Substituting (9) and (10) into (6), we obtain that

$$\begin{aligned} \left|Y_t^\delta\right| &\le C_5(\omega) + C_2(\omega)\int_0^t \left|Y_{t_u}^\delta\right| u^{-\alpha} du + C_4(\omega)\left(1+Y_t^{\delta,*}\right)\delta^{H-\alpha-\rho} \\ &\quad + C_6(\omega)\int_0^t \int_0^{t_r} \varphi_{r,u} du\, dr \end{aligned} \tag{11}$$

with $C_5(\omega) = C_3(\omega) + MC_0(\omega)C_2$, $C_6(\omega) = MC_0(\omega)$. To simplify the notations, in what follows we remove subscripts from $C(\omega)$, writing $C(\omega)$ for all constants depending on ω.

So we can write

$$Y_t^{\delta,*} \le C(\omega)\Big(1 + Y_t^{\delta,*}\delta^{H-\alpha-\rho} + \int_0^t \left|Y_{t_u}^\delta\right| u^{-\alpha} du + \int_0^t \int_0^{t_r} \varphi_{r,u} du\, dr\Big). \tag{12}$$

In turn, we can estimate $\int_0^{t_s} \varphi_{s,u} du$. At first, similarly to the previous estimates,

$$\begin{aligned} \left|Y_{t_s}^\delta - Y_u^\delta\right| &\le C(\omega)\Big[\int_u^{t_s}\Big(1+\left|Y_{t_v}^\delta\right|\Big)dv + \int_u^{t_s}\Big(1+\left|Y_{t_v}^\delta\right|\Big)(v-u)^{-\alpha}dv \\ &\quad + \int_u^{t_s}\int_u^{t_v}\left|\sigma(t_v, Y_{t_v}^\delta) - \sigma(t_z, Y_{t_z}^\delta)\right|(v-z)^{-\alpha-1}dz\, dv\Big] \\ &\le C(\omega)\Big[(t_s-u)^{1-\alpha} + \int_u^{t_s}\left|Y_{t_v}^\delta\right|(v-u)^{-\alpha}dv + \delta^\beta\int_u^{t_s}(v-t_v)^{-\alpha}dv \\ &\quad + \int_u^{t_s}\int_u^{t_v}\varphi_{v,z}dz\, dv + \int_u^{t_s}\int_u^{t_v}\left|Y_z^\delta - Y_{t_z}^\delta\right|(v-z)^{-\alpha-1}dz\, dv\Big]; \end{aligned} \tag{13}$$

multiplying by $(s-u)^{-\alpha-1}$ and integrating over $[0, t_s]$, we obtain that

$$\int_0^{t_s} \varphi_{s,u} du \le C(\omega)\sum_{i=1}^5 Q_s^i, \tag{14}$$

where

$$Q_s^1 := \int_0^{t_s} (t_s-u)^{1-\alpha}(s-u)^{-\alpha-1}du \le \int_0^{t_s}(s-u)^{-2\alpha}du \le C; \tag{15}$$

$$\begin{aligned} Q_s^2 &:= \int_0^{t_s}(s-u)^{-\alpha-1}\int_u^{t_s}\left|Y_{t_v}^\delta\right|(v-u)^{-\alpha}dv \\ &= \int_0^{t_s}\left|Y_{t_v}^\delta\right|\int_0^v (v-u)^{-\alpha}(s-u)^{-\alpha-1}du\,dv \le C_0\int_0^{t_s}\left|Y_{t_v}^\delta\right|(s-v)^{-2\alpha}dv, \end{aligned} \tag{16}$$

where $C_0 = \int_0^\infty (1+y)^{-\alpha-1}y^{-\alpha}dy$; according to Lemma 1

$$\begin{aligned} Q_s^3 &:= \delta^\beta\int_0^{t_s}(s-u)^{-\alpha-1}\int_u^{t_s}(v-t_v)^{-\alpha}dv\,du \\ &\le C\delta^\beta\delta^{-\alpha} \le C. \end{aligned} \tag{17}$$

Further, using estimates (7), we can conclude that

$$\begin{aligned} Q_s^4 &:= \int_0^{t_s}(s-u)^{-\alpha-1}\int_u^{t_s}\int_u^{t_v}\varphi_{v,z}dz\,dv\,du \\ &\le \int_0^{t_s}\int_0^{t_v}\int_0^{z\wedge v}\varphi_{v,z}(s-u)^{-\alpha-1}du\,dz\,dv \le C\int_0^{t_s}(s-v)^{-\alpha}\int_0^{t_v}\varphi_{v,z}dz\,dv. \end{aligned} \tag{18}$$

At last, using estimates (7) and Lemma 1, we can conclude that.

$$\begin{aligned} Q_s^5 &:= \int_0^{t_s}(s-u)^{-\alpha-1}\int_u^{t_s}\int_u^{t_v}\left|Y_z^\delta - Y_{t_z}^\delta\right|(v-z)^{-\alpha-1}dz\,dv\,du \\ &\le C(\omega)\int_0^{t_s}(s-u)^{-\alpha-1}\int_u^{t_s}\int_u^{t_v}(v-z)^{-\alpha-1}dz\,dv\,du\cdot\delta^{H-\rho}\Big(1+\left|Y_{t_s}^{\delta,*}\right|\Big) \\ &\le C(\omega)\Big(1+\left|Y_{t_s}^{\delta,*}\right|\Big)\delta^{H-\rho-\alpha}. \end{aligned} \tag{19}$$

Now, denote $\psi_s := Y_s^{\delta,*} + \int_0^{t_s}\varphi_{s,u}du$. Note that the integrals Q_s^i are finite for $s = k\delta$, i.e. for any $s\in[0,T]$, including $s = t_s$. Then it follows from (12) and (14)–(19) that

$$\psi_t \le C(\omega)\Big(1 + Y_t^{\delta,*}\delta^{H-\alpha-\rho} + \int_0^t\big((t-v)^{-2\alpha}+v^{-\alpha}\big)\psi_v dv\Big).$$

Let $\varepsilon>0$ be fixed. Note that all constants $C(\omega)$ are finite a.s. and independent of δ. Thus, we can choose $\delta_0>0$ and $\Omega_{\varepsilon,\delta_0,\rho}$ such that $C(\omega)\delta_0^{H-\alpha-\rho}\le 1/2$ on $\Omega_{\varepsilon,\delta_0,\rho}$ and $P(\Omega_{\varepsilon,\delta_0,\rho})>1-\varepsilon$. Then for any $\omega\in\Omega_{\varepsilon,\delta_0,\rho}$

$$\psi_t \le C(\omega) + \frac{1}{2}\psi_t + C(\omega)\int_0^t\big((t-v)^{-2\alpha}+v^{-\alpha}\big)\psi_v dv,$$

whence

$$\psi_t \le C(\omega)\Big(1 + t^{2\alpha}\int_0^t (t-v)^{-2\alpha}v^{-2\alpha}\psi_v dv\Big),$$

and it follows immediately from the last equation and Proposition 1 that $\psi_t\le C(\omega)$ whence, in particular, $\left|Y_t^\delta\right|\le C(\omega)$, $t\in[0,T]$, and $\int_0^{t_s}\varphi_u du \le C(\omega)$. Moreover,

from (13) with $u = t_r$, $r \le s$, taking into account that $\int_{t_r}^{t_s}(v-t_v)^{-\alpha}dv \le \delta^{-\alpha}(t_s-t_r)$, we obtain the estimate

$$\left|Y^\delta_{t_s} - Y^\delta_{t_r}\right| \le C(\omega)\Big((t_s - t_r)^{1-\alpha} + \delta^{\beta-\alpha}(t_s - t_r) + (t_s - t_r) \\ + \delta^{H-\rho}\int_{t_r}^{t_s}(v - t_v)^{-\alpha}dv\Big) \le C(\omega)(t_s - t_r)^{1-\alpha},$$

and the statement (i) is proved. (ii) Let $|b(t, x)| \le b$, $|\sigma(t, x)| \le \sigma$. Then it is very easy to see that the estimate (11) will take a form

$$\left|Y^\delta_t\right| \le C(\omega)\Big(1 + \int_0^t\int_0^{t_r}\varphi_{r,u}du\,dr\Big),$$

(13) will perform to

$$\left|Y^\delta_{t_s} - Y^\delta_u\right| \le C(\omega)\Big((t_s - u)^{1-\alpha} + (\delta^\beta + \delta^{H-\rho})\int_u^{t_s}(v - t_v)^{-\alpha}dv \\ + \int_u^{t_s}\int_u^{t_v}\varphi_{v,z}dz\,dv\Big)$$

and instead of (14)–(19) we obtain

$$\int_0^{t_s}\varphi_{s,u}du \le C(\omega)\Big(1 + \int_0^{t_s}(s - v)^{-\alpha}\int_0^{t_v}\varphi_{v,z}dz\,dv\Big),$$

whence the proof easily follows. □

3. The estimates of rate of convergence for Euler approximations of the solutions of pathwise equations

Now we establish the estimates of the rate of convergence of our approximations (4) for the solution of the equation (1) with pathwise integral w.r.t. fBm. We establish even more, namely, the estimate of convergence rate for the norm of the difference $X_t - Y^\delta_t$ in some Besov space, similarly to the result of Theorem 1. Denote

$$\Delta_{u,s}(X, Y^\delta) := \left|X_s - Y^\delta_s - X_u + Y^\delta_u\right|$$

and assume for technical simplicity that $L_N = L$, $M_N = M$ in (A) and (B).

Theorem 2. *Let the conditions* (A)–(C) *hold and also*

(D) 1) *Hölder continuity of the coefficient b in time:* $|b(t, x) - b(s, x)| \le C|t - s|^\gamma$, $C > 0$, $2H - 1 < \gamma \le 1$*;*
2) *the exponent β from* (A) 3) *satisfies $\beta > H$.*

Then:

(i) for any $\varepsilon > 0$ and any $\rho > 0$ sufficiently small there exists $\delta_0 > 0$ and $\Omega_{\varepsilon,\delta_0,\rho}$ such that $P(\Omega_{\varepsilon,\delta_0,\rho}) > 1 - \varepsilon$ and for any $\omega \in \Omega_{\varepsilon,\delta_0,\rho}$, $\delta < \delta_0$

$$U_\delta := \sup_{0\le s\le T}\Big(\left|X_s - Y^\delta_s\right| + \int_0^{t_s}\left|\Delta_{u,s}(X, Y^\delta)\right|(s - u)^{-\alpha-1}du\Big) \le C(\omega)\cdot\delta^{2H-1-\rho},$$

where $C(\omega)$ does not depend on δ and ε (but depends on ρ);

(ii) if, in addition, the coefficients b and σ are bounded, then for any $\rho \in (0, 2H-1)$ there exists $C(\omega) < \infty$ a.s. such that $U_\delta \le C(\omega)\delta^{2H-1-\rho}$, $C(\omega)$ does not depend on δ.

Proof. (i) Denote $Z_t^\delta := \sup_{0\le s\le t}\left|X_s - Y_s^\delta\right|$. Then

(20)
$$\begin{aligned}
&Z_t^\delta := \sup_{0\le s\le t}|X_s - Y_s^\delta| \le \sup_{0\le s\le t}\int_0^s |b(u,X_u) - b(t_u, Y_{t_u}^\delta)|du\\
&+ \sup_{0\le s\le t}\sum_{i,j=1}^m \Big|\int_0^s (\sigma^{ij}(u,X_u) - \sigma^{ij}(t_u, Y_{t_u}^\delta))dB_u^i\Big| \le \int_0^t |b(u,X_u) - b(u,Y_u^\delta)|du\\
&+ \int_0^t |b(u,Y_u^\delta) - b(t_u,Y_u^\delta)|du + \int_0^t |b(t_u,Y_u^\delta) - b(t_u,Y_{t_u}^\delta)|du\\
&+ \sup_{0\le s\le t}\sum_{i,j=1}^m \Big|\int_0^s (\sigma^{ij}(u,X_u) - \sigma^{ij}(u,Y_u^\delta))dB_u^i\Big|\\
&+ \sup_{0\le s\le t}\sum_{i,j=1}^m \Big|\int_0^s (\sigma^{ij}(u,Y_u^\delta) - \sigma^{ij}(t_u,Y_u^\delta))dB_u^i\Big|\\
&+ \sup_{0\le s\le t}\sum_{i,j=1}^m \Big|\int_0^s (\sigma^{ij}(t_u,Y_u^\delta) - \sigma^{ij}(t_u,Y_{t_u}^\delta))dB_u^i\Big| =: \sum_{k=1}^6 I_k.
\end{aligned}$$

Now we estimate separately all these terms. Evidently,

$$I_1 \le L\int_0^t Z_u^\delta du. \tag{21}$$

Condition (D) 1) implies that for $\delta \le 1$

$$I_2 \le C\int_0^t |u - t_u|^\gamma\, du \le C\delta^\gamma \le C\delta^{2H-1}. \tag{22}$$

As it follow from Theorem 2.2, for any $\varepsilon > 0$ and any $\rho \in (0,H)$ there exists $\delta_0 > 0$ and $\Omega_{\varepsilon,\delta_0,\rho} \subset \Omega$ such that $P(\Omega_{\varepsilon,\delta_0,\rho}) > 1-\varepsilon$ and $C(\omega)$ independent of ε and δ such that for any $\omega \in \Omega_{\varepsilon,\delta_0,\rho}$ it holds $\left|Y_t^\delta - Y_s^\delta\right| \le C(\omega)\left|t-s\right|^{H-\rho}$. In what follows we assume that $\delta < \delta_0 < 1$. Therefore

$$I_3 \le L\cdot C(\omega)\delta^{H-\rho}\cdot t \le C(\omega)\delta^{H-\rho}, \omega \in \Omega_{\varepsilon,\delta_0,\rho}. \tag{23}$$

Now we go on with I_4. It follows from (2) that for $1-H < \alpha < 1/2$

$$\begin{aligned}
I_4 \le C(\omega)\sum_{i,j=1}^m \Big[\int_0^t &\left|\sigma^{ij}(u,X_u) - \sigma^{ij}(u,Y_{t_u}^\delta)\right| u^{-\alpha}du\\
+ \int_0^t\int_0^r &\left|\sigma^{ij}(r,X_r) - \sigma^{ij}(u,X_u) - \sigma^{ij}(r,Y_r^\delta) + \sigma^{ij}(u,Y_u^\delta)\right|\\
&\times (r-u)^{-\alpha-1}du\,dr\Big] =: I_7 + I_8.
\end{aligned} \tag{24}$$

Evidently,

$$I_7 \le C(\omega)\int_0^t Z_u^\delta u^{-\alpha}du. \tag{25}$$

According to [11, Lemma 7.1], under condition (A)

$$\begin{aligned} |\sigma(t_1,x_1)-\sigma(t_2,x_2)-\sigma(t_1,x_3)+\sigma(t_2,x_4)| &\le M\,|x_1-x_2-x_3+x_4| \\ &+ M\,|x_1-x_3|\Big(|t_2-t_1|^{\beta}+|x_1-x_2|^{\kappa}+|x_3-x_4|^{\kappa}\Big). \end{aligned} \tag{26}$$

Therefore, $I_8\le\sum_{k=9}^{12}I_k$, where

$$\begin{aligned} I_9&=C(\omega)\int_0^t\int_0^r\left|X_r-Y^\delta_r\right|(r-u)^{\beta-\alpha-1}du\,dr,\\ I_{10}&=C(\omega)\int_0^t\int_0^r\left|X_r-Y^\delta_r\right|\left|X_r-X_u\right|^{\kappa}(r-u)^{-\alpha-1}du\,dr,\\ I_{11}&=C(\omega)\int_0^t\int_0^r\left|X_r-Y^\delta_r\right|\left|Y^\delta_r-Y^\delta_u\right|^{\kappa}(r-u)^{-\alpha-1}du\,dr,\\ I_{12}&=C(\omega)\int_0^t\int_0^r\Delta_{u,r}(X,Y^\delta)(r-u)^{-\alpha-1}du\,dr. \end{aligned}$$

Taking into account that $\beta>H>\alpha$, we obtain that

$$I_9\le C(\omega)\int_0^t Z^\delta_u du. \tag{27}$$

As it follows from [11, Theorem 2.1], under assumptions (A) and (B) for any $0<\rho<H$ there exists such constant $C(\omega)$ that

$$\sup_{0\le t\le T}|X_t|\le C(\omega),\qquad \sup_{0\le s\le t\le T}|X_t-X_s|\le C(\omega)\,|t-s|^{H-\rho}. \tag{28}$$

Moreover, we can choose $\rho>0$ and $\alpha>1-H$ such that $\kappa(H-\rho)>\alpha$ and $H-\rho>2H-1$, because $\kappa H>1-H$. In this case

$$I_{10}\le C(\omega)\int_0^t Z^\delta_r\int_0^r(r-u)^{\kappa(H-\rho)-\alpha-1}du\,dr\le C(\omega)\int_0^T Z^\delta_r dr. \tag{29}$$

It follows from Theorem 2.2 that on $\Omega_{\varepsilon,\delta_0,\rho}$ the same estimate holds for I_{11}.

Now estimate I_5.

$$\begin{aligned} I_5&\le C(\omega)\int_0^t\left|\sigma(u,Y^\delta_u)-\sigma(t_u,Y^\delta_u)\right|u^{-\alpha}du\\ &+C(\omega)\int_0^t\int_0^r\left|\sigma(r,Y^\delta_r)-\sigma(t_r,Y^\delta_r)-\sigma(u,Y^\delta_u)+\sigma(t_u,Y^\delta_u)\right|(r-u)^{-\alpha-1}du\,dr\\ &=: I_{13}+I_{14}. \end{aligned}$$

Obviously,

(30)

$$\begin{aligned} I_{13}&\le C(\omega)\delta^{\beta},\\ I_{14}&\le C(\omega)\Big(\int_0^t\int_0^{t_r}+\int_0^t\int_{t_r}^r\Big)|...|\,(r-u)^{-\alpha-1}du\,dr\\ &\le C(\omega)\int_0^t\int_0^{t_r}\delta^{\beta}(r-u)^{-\alpha-1}du\,dr+\int_0^t\int_{t_r}^r\big((r-u)^{\beta}+(r-u)^{H-\rho}\big)du\,dr\\ &\le C(\omega)\big(\delta^{\beta-\alpha}+\delta^{H-\rho-\alpha}\big). \end{aligned} \tag{31}$$

Similarly,

$$
\begin{aligned}
I_6 &\le C(\omega)\int_0^t \left|\sigma(t_u, Y_u^\delta) - \sigma(t_u, Y_{t_u}^\delta)\right| u^{-\alpha} du \\
&+ C(\omega)\int_0^t\int_0^r \left|\sigma(t_r, Y_r^\delta) - \sigma(t_r, Y_{t_r}^\delta) - \sigma(t_u, Y_u^\delta) + \sigma(t_u, Y_{t_u}^\delta)\right| \\
&\qquad \times (r-u)^{-\alpha-1} du\, dr =: I_{15} + I_{16}.
\end{aligned} \tag{32}
$$

Here

$$I_{15} \le C(\omega)\int_0^t \delta^{H-\rho} u^{-\alpha} du \le C(\omega)\delta^{H-\rho}; \tag{33}$$

$$I_{16} \le C(\omega)\int_0^t\int_0^r \delta^{H-\rho}(r-u)^{-\alpha-1} du\, dr \le C(\omega)\delta^{H-\rho-\alpha}. \tag{34}$$

Substituting (21)–(34) into (20), we obtain that on $\Omega_{\varepsilon,\delta_0,\rho}$

$$Z_t^\delta \le C(\omega)\Big(\int_0^t Z_r^\delta r^{-\alpha} dr + \delta^{H-\rho-\alpha} + \delta^{H-\rho} + \int_0^t \theta_r dr\Big), \tag{35}$$

where $\theta_r = \int_0^r \Delta_{r,u}(X, Y^\delta)(r-u)^{-\alpha-1} du$. Recall that $H - \rho > 2H - 1$, therefore

$$Z_t^\delta \le C(\omega)\Big(\int_0^t \big(Z_r^\delta r^{-\alpha} + \theta_r\big) dr + \delta^{2H-1-\rho}\Big).$$

We now estimate θ_r. Evidently, for $t > u$

$$
\begin{aligned}
\Delta_{t,u}(X, Y^\delta) \le &\int_u^t \left|b(s, X_s) - b(t_s, Y_{t_s}^\delta)\right| ds \\
&+ \sum_{i,j=1}^m \left|\int_u^t \big(\sigma^{ij}(s, X_s) - \sigma^{ij}(t_s, Y_{t_s}^\delta)\big) dB_s^i\right|.
\end{aligned}
$$

Therefore, using inequality (2), we obtain that $\theta_t \le \sum_{k=1}^{9} J_k$, where

$$\begin{aligned}
J_1 &= \int_0^t \int_u^t \left| b(s,X_s) - b(s,Y_s^\delta)\right| ds (t-u)^{-\alpha-1} du,\\
J_2 &= \int_0^t \int_u^t \left| b(s,Y_s^\delta) - b(t_s,Y_s^\delta)\right| ds (t-u)^{-\alpha-1} du,\\
J_3 &= \int_0^t \int_u^t \left| b(t_s,Y_s^\delta) - b(t_s,Y_{t_s}^\delta)\right| ds (t-u)^{-\alpha-1} du,\\
J_4 &= C(\omega)\int_0^t \int_u^t \left| \sigma(s,X_s) - \sigma(s,Y_s^\delta)\right| (s-u)^{-\alpha} ds (t-u)^{-\alpha-1} du,\\
J_5 &= C(\omega)\int_0^t \int_u^t \left| \sigma(s,Y_s^\delta) - \sigma(t_s,Y_s^\delta)\right| (s-u)^{-\alpha} ds (t-u)^{-\alpha-1} du,\\
J_6 &= C(\omega)\int_0^t \int_u^t \left| \sigma(t_s,Y_s^\delta) - \sigma(t_s,Y_{t_s}^\delta)\right| (s-u)^{-\alpha} ds (t-u)^{-\alpha-1} du,\\
J_7 &= C(\omega)\int_0^t \int_u^t \int_u^r \left| \sigma(r,X_r) - \sigma(r,Y_r^\delta) - \sigma(v,X_v) + \sigma(v,Y_v^\delta)\right|\\
&\qquad \times (r-v)^{-\alpha-1} dv\, dr (t-u)^{-\alpha-1} du,\\
J_8 &= C(\omega)\int_0^t \int_u^t \int_u^r \left| \sigma(r,Y_r^\delta) - \sigma(t_r,Y_r^\delta) - \sigma(v,Y_v^\delta) + \sigma(t_v,Y_v^\delta)\right|\\
&\qquad \times (r-v)^{-\alpha-1} dv\, dr (t-u)^{-\alpha-1} du,\\
J_9 &= C(\omega)\int_0^t \int_u^t \int_u^r \left| \sigma(t_r,Y_r^\delta) - \sigma(t_r,Y_{t_r}^\delta) - \sigma(t_v,Y_v^\delta) + \sigma(t_v,Y_{t_v}^\delta)\right|\\
&\qquad \times (r-v)^{-\alpha-1} dv\, dr (t-u)^{-\alpha-1} du.
\end{aligned}$$

It is clear that

$$J_1 \le C\int_0^t Z_s^\delta \int_0^s (t-u)^{-\alpha-1} du\, ds, \quad J_2 \le C\delta^\gamma, \quad J_3 \le C(\omega)\delta^{H-\rho}.$$

Further,

$$J_4 \le C\int_0^t Z_s^\delta \int_0^s (s-u)^{-\alpha}(t-u)^{-\alpha-1} du\, ds.$$

As we noted before, the inner integral $\int_0^s (s-u)^{-\alpha}(t-u)^{-\alpha-1}du \le C_0(t-s)^{-2\alpha}$, $C_0 = \int_0^\infty (1+y)^{-\alpha-1}y^{-\alpha}dy$. Therefore

$$J_4 \le C\int_0^t (t-s)^{-2\alpha} Z_s^\delta ds.$$

Similarly to J_2, $J_5 \le C(\omega)\delta^\gamma$, and similarly to J_3, $J_6 \le C(\omega) \le C(\omega)\delta^{H-\rho}$. Estimating J_7, J_8 and J_9 is, of course, a bit more complicated, but not dramatically. Obviously,

$$\begin{aligned}
J_8 &\le C(\omega)\delta^\beta \int_0^t \int_u^t \int_u^r (r-v)^{-\alpha-1} dv\, dr (t-u)^{-\alpha-1} du\\
&= C(\omega)\delta^\beta \int_0^t (t-u)^{-2\alpha} du \le C(\omega)\delta^\beta;
\end{aligned}$$

similarly $J_9 \le C(\omega)\delta^{H-\rho}$. Now we apply to J_7 the inequality (26) and obtain the following estimate of the integrand:

$$\begin{aligned} &\left|\sigma(r, X_r) - \sigma(r, Y_r^\delta) - \sigma(v, X_v) + \sigma(v, Y_v^\delta)\right| \le M\Big[\Delta_{r,v}(X, Y^\delta) \\ &+ \left|X_r - Y_r^\delta\right|(r-v)^\beta + \left|X_r - Y_r^\delta\right|\left|X_r - X_v\right|^\kappa + \left|X_r - Y_r^\delta\right|\left|Y_r^\delta - Y_v^\delta\right|^\kappa\Big]. \end{aligned} \tag{36}$$

According to this, we write $J_7 \le \sum_{k=10}^{13} J_k$, where, in turn,

$$\begin{aligned} J_{10} &= C(\omega)\int_0^t\int_u^t\int_u^r \Delta_{r,v}(X, Y^\delta)(r-v)^{-\alpha-1}dv\,dr(t-u)^{-\alpha-1}du \\ &= C(\omega)\int_0^t\int_0^r\int_0^v (t-u)^{-\alpha-1}du\Delta_{r,v}(X, Y^\delta)(r-v)^{-\alpha-1}dr\,dv \\ &\le C(\omega)\int_0^t (t-r)^{-\alpha}\theta_r dr; \\ J_{11} &= C(\omega)\int_0^t\int_u^t\int_u^r \left|X_r - Y_r^\delta\right|(r-v)^{\beta-\alpha-1}dv\,dr(t-u)^{-\alpha-1}du \\ &\le C(\omega)\int_0^t Z_r^\delta\int_0^r (t-u)^{-\alpha-1}\Big(\int_u^r (r-v)^{\beta-\alpha-1}dv\Big)du\,dr \\ &\le C(\omega)\int_0^t (t-r)^{-\alpha}Z_r dr, \\ J_{12} &= C(\omega)\int_0^t\int_u^t\int_u^r \left|X_r - Y_r^\delta\right|\left|X_r - X_v\right|^\kappa (r-v)^{-\alpha-1}dv\,dr(t-u)^{-\alpha-1}du \\ &\le C(\omega)\int_0^t\int_0^r\int_u^r Z_r^\delta(r-v)^{\kappa(H-\rho)-\alpha-1}dv(t-u)^{-\alpha-1}du\,dr \\ &\le C(\omega)\int_0^t Z_r^\delta(t-r)^{-\alpha}dr, \end{aligned}$$

and $J_{13} \le C(\omega)\int_0^t Z_r^\delta(t-r)^{-\alpha}dr$ is obtained the same way. Summing up these estimates, we obtain that

$$J_7 \le C(\omega)\int_0^t (t-r)^{-\alpha}\left(Z_r^\delta + \theta_r\right)dr,$$

whence

$$\theta_t \le C(\omega)\Big(\int_0^t (t-r)^{-2\alpha}\left(Z_r^\delta + \theta_r\right)dr + \delta^{H-\rho} + \delta^\gamma\Big). \tag{37}$$

Coupling together (35) and (37), and taking into account that $H-\rho > 2H-1$, $\gamma > 2H-1$, we obtain

$$\begin{aligned} Z_t^\delta + \theta_t &\le C(\omega)\Big(\delta^{2H-1} + \int_0^t \left((t-r)^{-2\alpha} + r^{-\alpha}\right)\left(Z_r^\delta + \theta_r\right)dr\Big) \\ &\le C(\omega)\Big(\delta^{2H-1} + t^{2\alpha}\int_0^t (t-r)^{-2\alpha}r^{-2\alpha}\left(Z_r^\delta + \theta_r\right)dr\Big) \end{aligned} \tag{38}$$

The proof now follows immediately from (38) and Proposition 2.1.

The statement (ii) is obvious. □

Remark 1. In [9] it is proved that for an equation with homogeneous regular coefficients the error $\left|X_t - Y_t^\delta\right| \delta^{1-2H}$ almost surely converges to some stochastic process ξ_t, which means that the estimate of the rate of convergence in Theorem 2 is sharp.

4. Approximation of quasilinear Skorohod-type equations

Now we proceed to the problem of numerical solution of Skorohod-type equation driven by fractional white noise. From now on, we assume that our probability space is the white noise space $(\Omega, \mathcal{F}, P) = (S'(R), \mathcal{B}(S'(R)), \mu)$, $\diamond$ is the Wick product, $B_t^0 = \langle \omega, \mathbf{1}_{[0,t]} \rangle$ is Brownian motion, $W^0 = \dot{B}^0$ is the white noise (see [4] for definitions). Next, in order to introduce an fBm with Hurst parameter $H > 1/2$ on this space, we define for $f : [0,T] \to R$ the fractional integral operator

$$Mf(x) = K \int_x^T (s-x)^{H-3/2} f(s)\, ds,$$

where

$$K = \big(\sin(\pi H)\Gamma(2H+1)\big)^{1/2} \big(K_1^2 + K_2^2\big)^{-1/2},$$
$$K_1 = \pi\Big(2\cos((3/4 - H/2)\pi)\Gamma(3/2 - H)\Big)^{-1},$$
$$K_2 = \pi\Big(2\sin((3/4 - H/2)\pi)\Gamma(3/2 - H)\Big)^{-1},$$

and set $M_t(x) = M\mathbf{1}_{[0,t]}(x)$. We also define for $f, g : [0,T] \to R$ the scalar product and the norm

$$\langle f, g \rangle_{\mathcal{H}} = H(2H-1) \int_0^T \int_0^T f(t) g(s) \left|t-s\right|^{2H-2} dt\, ds, \quad \|f\|_{\mathcal{H}}^2 = \langle f, f \rangle_{\mathcal{H}}.$$

The process

$$B_t = \langle M_t, \omega \rangle, \quad t \in [0,T]$$

is the fBm with Hurst parameter H. Let also $W = \dot{B}$ be the fractional white noise. Detailed description of the white noise theory can be found in [2], [5].

Consider quasilinear Skorohod-type equation driven by fractional white noise

$$X(t) = X_0 + \int_0^t b(s, X(s), \omega)\, ds + \int_0^t \sigma(s) X(s) \diamond W(s)\, ds \tag{39}$$

with non-random initial condition X_0. Suppose that coefficients b and σ satisfy the following conditions:

(E) 1) The linear growth condition and Lipschitz condition on b:

$$|b(t,x,\omega)| \le C(1+|x|),$$
$$|b(t,x,\omega) - b(t,y,\omega)| \le C\left|x-y\right|;$$

2) "Smoothness" of b w.r.t. ω: for any $t \in [0,T]$ and for $h \in L^1(R)$

$$|b(t,x,\omega+h) - b(t,x,\omega)| \le C(1+|x|) \int_R |h(s)|\, ds.$$

3) Hölder continuity of b w.r.t. t or order H with constant that grows linearly in x:

$$|b(t,x,\omega) - b(s,x,\omega)| \le C(1+|x|)\left|t-s\right|^H;$$

4) Hölder continuity of σ w.r.t. t or order H:

$$|\sigma(t)-\sigma(s)| \le C\,|t-s|^H .$$

Remark 2. As in previous sections, we denote by C any constant which may depend on coefficients of the equation, on initial condition X_0 and on the time horizon T, but is independent of anything else (and we write $C(v)$ to emphasize the dependence on v).

Remark 3. The condition (E) 2) is true if, for example, the coefficient b has stochastic derivative growing at most linearly in x. It is obviously true if b is non-random.

Define for $t\in[0,T]$ $\sigma_t(s)=\sigma(s)\mathbf{1}_{[0,t]}(s)$ and denote

$$J_\sigma(t)=\exp^\diamond\left\{-\int_0^t \sigma(s)dB_s\right\}=\exp\left\{-\int_R M\sigma_t(s)dB^0(s)-\frac{1}{2}\left\|\sigma_t\right\|^2_{\mathcal{H}}ds\right\}$$

the fractional Wick exponent. It follows from [8, Theorem 2] that under assumptions (E) equation (39) has the unique solution that belongs to all L^p and can be represented in the form

$$X(t)=J_{-\sigma}(t)\diamond Z(t),$$

where the process $Z(t)$ solves (ordinary) differential equation

$$Z(t)=X_0+\int_0^t J_\sigma(s)b(s,J_\sigma^{-1}(s)Z(s),\omega+M\sigma_s)\,ds. \tag{40}$$

This gives the following idea of constructing time-discrete approximations of the solution of (39). Take the uniform partitioning $\{\tau_n=n\delta,\ n=1,\dots,N\}$ of the segment $[0,T]$ and define first the approximations of Z in a recursive way:

$$\begin{aligned}&\widetilde{Z}(0)=X_0,\\&\widetilde{Z}(\tau_{n+1})=\widetilde{Z}(\tau_n)+\widetilde{J}(\tau_n)b(\tau_n,\widetilde{J}^{-1}(\tau_n)\widetilde{Z}(\tau_n),\omega+M\widetilde{\sigma}_n)\delta,\end{aligned} \tag{41}$$

where

$$\begin{aligned}&\widetilde{J}(t)=\exp\left\{-\int_0^t\widetilde{\sigma}(s)dB_s-\frac{1}{2}\left\|\widetilde{\sigma}\mathbf{1}_{[0,t]}\right\|^2_{\mathcal{H}}\right\},\\&\widetilde{\sigma}(s)=\sigma(t_s),\ \widetilde{\sigma}_n=\widetilde{\sigma}\mathbf{1}_{[0,\tau_n]}.\end{aligned}$$

Note that both $\|\widetilde{\sigma}_n\|_{\mathcal{H}}$ and $M\widetilde{\sigma}_n$ are easily computable as finite sums of elementary integrals. Further, we interpolate continuously by

$$\widetilde{Z}(t)=X_0+\int_0^t\widetilde{J}(t_s)b(t_s,\widetilde{J}^{-1}(t_s)\widetilde{Z}(t_s),\omega+M\widetilde{\sigma}_{n_s})\,ds, \tag{42}$$

where $n_s=\max\{n:\tau_n\le s\}$, and set

$$\widetilde{X}(t)=T_{-M\widetilde{\sigma}\mathbf{1}_{[0,t]}}\widetilde{J}^{-1}(t)\widetilde{Z}(t), \tag{43}$$

where for $h\in S'(R)$ T_h is the shift operator, $T_hF(\omega)=F(\omega+h)$.

Lemma 2. *Under the assumption* (E) 1) *the following estimate is true*

$$\left|e^{\alpha_1}b(t,e^{-\alpha_1}x,\omega)-e^{\alpha_2}b(t,e^{-\alpha_2}x,\omega)\right|\le C(1+e^{\alpha_1}+e^{\alpha_2}+|x|)\,|\alpha_1-\alpha_2|\,.$$

Proof. Write

$$\begin{aligned}&\left|e^{\alpha_1}b(t,e^{-\alpha_1}x,\omega)-e^{\alpha_2}b(t,e^{-\alpha_2}x,\omega)\right|\\&\le\left|e^{\alpha_1}b(t,e^{-\alpha_1}x,\omega)-e^{\alpha_1}b(t,e^{-\alpha_2}x,\omega)\right|+\left|e^{\alpha_1}b(t,e^{-\alpha_2}x,\omega)-e^{\alpha_2}b(t,e^{-\alpha_2}x,\omega)\right|\end{aligned}$$

and apply (E) 1). □

Lemma 3. *Let ξ_1 and ξ_2 be jointly Gaussian variables. Then for $q \geq 1$*

$$\mathsf{E}\left[\left|e^{\xi_1} - e^{\xi_2}\right|^{2q}\right] \leq C(L,q)\left(\mathsf{E}\left[(\xi_1-\xi_2)^2\right]\right)^q,$$

where $L = \max\left\{\mathsf{E}\left[\xi_1^2\right], \mathsf{E}\left[\xi_2^2\right]\right\}$.

Proof. By Lagrange theorem, Cauchy–Schwartz inequality and Gaussian property,

$$\begin{aligned}\mathsf{E}\left[\left|e^{\xi_1} - e^{\xi_2}\right|^{2q}\right] &\leq \left(\mathsf{E}\left[e^{4q\xi_1} + e^{4q\xi_2}\right]\mathsf{E}\left[|\xi_1-\xi_2|^{4q}\right]\right)^{1/2} \\ &\leq C(L)C(q)\left(\mathsf{E}\left[(\xi_1-\xi_2)^2\right]\right)^q,\end{aligned}$$

as required. □

Our first result is about convergence of $\widetilde{Z}$ to Z.

Theorem 3. *Under conditions* (E) *for any $p \geq 1$ the following estimate holds:*

$$\mathsf{E}\left[\left|Z(t) - \widetilde{Z}(t)\right|^{2p}\right] \leq C(p)\delta^{2pH}. \tag{44}$$

Proof. Firstly, we remind that $Z(t)$ belongs to all L^q and $\mathsf{E}\left[|Z(t)|^q\right] \leq C(q)$. Therefore equation (40) together with the condition (E) 2) gives $\mathsf{E}\left[|Z(t)-Z(s)|^q\right] \leq C(q)\left|t-s\right|^q$. Equation (41) and the condition (E) 1) allow to write

$$\left|\widetilde{Z}(\tau_{n+1})\right| \leq (1+C\delta)\left|\widetilde{Z}(\tau_n)\right| + C\delta\widetilde{J}(\tau_n) \leq e^{C\delta}\left|\widetilde{Z}(\tau_n)\right| + C\delta\widetilde{J}(\tau_n).$$

This gives an estimate

$$\left|\widetilde{Z}(\tau_n)\right| \leq C\sum_{k=0}^{N-1}\widetilde{J}(\tau_k)\delta.$$

Then for any $q \geq 1$ by the Jensen inequality,

$$\left|\widetilde{Z}(\tau_n)\right|^q \leq C(q)\sum_{k=0}^{N-1}\widetilde{J}^q(\tau_k)\delta,$$

Taking expectations, we get

$$\mathsf{E}\left[\left|\widetilde{Z}(\tau_n)\right|^q\right] \leq C(q)\sum_{k=0}^{N-1}\mathsf{E}\left[\widetilde{J}^q(\tau_k)\right]\delta.$$

Using that each $\widetilde{J}$ is exponent of Gaussian variable and σ is bounded on $[0,T]$, we obtain

$$\mathsf{E}\left[\left|\widetilde{Z}(\tau_n)\right|^q\right] \leq C(q)\sum_{k=0}^{N-1}\delta = C(q).$$

This through (42) and (E) 1) implies $\mathsf{E}\left[\left|\widetilde{Z}(t)\right|^q\right] \leq C(q)$.

Now write

$$\left|Z(t) - \widetilde{Z}(t)\right| \leq I_1 + I_2 + I_3 + I_4 + I_5,$$

where

$$I_1 = \left| \int_0^t \widetilde{J}(t_s)\big(b(t_s, \widetilde{J}^{-1}(t_s)Z(t_s), \omega + M\widetilde{\sigma}_{n_s}) - b(t_s, \widetilde{J}^{-1}(t_s)\widetilde{Z}(t_s), \omega + M\widetilde{\sigma}_{n_s})\big)\, ds \right|,$$

$$I_2 = \left| \int_0^t \big(\widetilde{J}(t_s)b(t_s, \widetilde{J}^{-1}(t_s)Z(t_s), \omega + M\widetilde{\sigma}_{n_s}) - J_\sigma(s)b(t_s, J_\sigma^{-1}(s)Z(t_s), \omega + M\widetilde{\sigma}_{n_s})\big)\, ds \right|,$$

$$I_3 = \left| \int_0^t J_\sigma(s)\big(b(s, J_\sigma^{-1}(s)Z(t_s), \omega + M\widetilde{\sigma}_{n_s}) - b(t_s, J_\sigma^{-1}(s)Z(t_s), \omega + M\widetilde{\sigma}_{n_s})\big)\, ds \right|,$$

$$I_4 = \left| \int_0^t J_\sigma(s)\big(b(s, J_\sigma^{-1}(s)Z(t_s), \omega + M\widetilde{\sigma}_{n_s}) - b(s, J_\sigma^{-1}(s)Z(t_s), \omega + M\sigma_s)\big)\, ds \right|,$$

$$I_5 = \left| \int_0^t J_\sigma(s)\big(b(s, J_\sigma^{-1}(s)Z(s), \omega + M\sigma_s) - b(s, J_\sigma^{-1}(s)Z(t_s), \omega + M\sigma_s)\big)\, ds \right|.$$

We first estimate using Lemma 2

$$\begin{aligned} I_2 &\le C \int_0^t \big(1 + J_\sigma(s) + \widetilde{J}(t_s) + |Z(t_s)|\big) \left(\left| \int_0^s \big(\sigma(u) - \widetilde{\sigma}(u)\big)\, dB_u \right| \right. \\ &\qquad \left. + \left|\sigma(t_s)\big(B_s - B(t_s)\big)\right| + \frac{1}{2}\left| \|\sigma_s\|^2_{\mathcal{H}} - \|\widetilde{\sigma}_{n_s}\|^2_{\mathcal{H}} \right| \right) ds \\ &\le C \int_0^t \big(1 + J_\sigma(s) + \widetilde{J}(t_s) + |Z(t_s)|\big) \\ &\qquad \cdot \left(\left| \int_0^s \big(\sigma(u) - \widetilde{\sigma}(u)\big)\, dB_u \right| + |B_s - B_{t_s}| + \delta^H \right) ds, \end{aligned}$$

where the inequality $\left| \|\sigma_s\|^2_{\mathcal{H}} - \|\widetilde{\sigma}_{n_s}\|^2_{\mathcal{H}} \right| < C\delta^H$ is due to E 4) and boundedness of σ on $[0,T]$. Applying Cauchy–Schwartz inequality, we arrive to

$$\begin{aligned} I_2 \le & C \left(\int_0^T \big(1 + J_\sigma^2(s) + \widetilde{J}^2(t_s) + Z^2(t_s)\big) ds \right)^{1/2} \\ & \cdot \left(\int_0^T \left(\left(\int_0^s \big(\sigma(u) - \widetilde{\sigma}(u)\big)\, dB_u \right)^2 + (B_t - B_{t_s})^2 + \delta^{2H} \right) ds \right)^{1/2}. \end{aligned}$$

Further, from (E) 3)

$$I_3 \le C \int_0^T (J_\sigma(s) + |Z(s)|)\, ds \delta^H,$$

from (E) 2)

$$I_3 \le C \int_0^T (J_\sigma(s) + |Z(s)|)\, ds \delta^H.$$

Condition (E) 1) allows to estimate

$$I_1 \le C \int_0^t \left| Z(t_s) - \widetilde{Z}(t_s) \right| ds,$$

$$I_5 \le C \int_0^t |Z(s) - Z(t_s)| \, ds.$$

Summing up these estimates yields

$$\begin{aligned}\left| Z(t) - \widetilde{Z}(t) \right| &\le C \left(\int_0^T \left(1 + J_\sigma^2(s) + \widetilde{J}^2(t_s) + Z^2(t_s)\right) ds \right)^{1/2} \\ &\cdot \left(\delta^{2H} + \int_0^T \left(\left(\int_0^s \left(\sigma(u) - \widetilde{\sigma}(u)\right) dB_u \right)^2 + (B_t - B_{t_s})^2 \right) ds \right)^{1/2} \\ &+ C \int_0^T \left| Z(t_s) - \widetilde{Z}(t_s) \right| ds + C \int_0^t |Z(s) - Z(t_s)| \, ds.\end{aligned}$$

Then, using (discrete) Gronwall inequality, we get

$$\begin{aligned}\left| Z(t) - \widetilde{Z}(t) \right| &\le C \left(\int_0^T \left(1 + J_\sigma^2(s) + \widetilde{J}^2(t_s) + Z^2(t_s)\right) ds \right)^{1/2} \\ &\cdot \left(\delta^{2H} + \int_0^T \left(\left(\int_0^s \left(\sigma(u) - \widetilde{\sigma}(u)\right) dB_u \right)^2 + (B_t - B_{t_s})^2 \right) ds \right)^{1/2} \\ &+ C \int_0^t |Z(s) - Z(t_s)| \, ds.\end{aligned}$$

Then we raise this to the $2p$th power and use Jensen's inequality. The last term will be bounded by $C(p)\delta^{2p}$, in the first one we apply Cauchy–Schwartz inequality for expectations, Jensen's inequality and use uniform boundedness of moments for Z, J_σ and $\widetilde{J}$ (for J_σ and $\widetilde{J}$ it follows from the fact that the both are exponents of some Gaussian variables with bonded variance) to get

$$\begin{aligned}\mathsf{E}\left[\left| Z(t) - \widetilde{Z}(t) \right|^{2p} \right] \le C(p) \Bigg(\delta^{2pH} &+ \left(\mathsf{E}\left[\left| \int_0^T \left(\sigma(u) - \widetilde{\sigma}(u)\right) dB_u \right|^{4p} \right] \right)^{1/2} \\ &+ \left(\mathsf{E}\left[|B_t - B_{t_s}|^{4p} \right] \right)^{1/2} \Bigg).\end{aligned}$$

Using again that $\mathsf{E}\left[|\cdot|^{4p} \right] = C(p)(\mathsf{E}\left[(\cdot)^2 \right])^{2p}$ for Gaussian variables, we get

$$\begin{aligned}\mathsf{E}\left[\left| Z(t) - \widetilde{Z}(t) \right|^{2p} \right] &\le C(p) \Bigg(\delta^{2pH} + \left(\mathsf{E}\left[\left| \int_0^T \left(\sigma(u) - \widetilde{\sigma}(u)\right) dB_u \right|^{2} \right] \right)^{p} \\ &\qquad + \left(\mathsf{E}\left[|B_t - B_{t_s}|^{2} \right] \right)^{p} \Bigg) \\ &\le C(p)\left(\delta^{2pH} + \|\sigma - \widetilde{\sigma}\|_{\mathcal{H}}^{2p}\right) \le C(p)\delta^{2pH},\end{aligned}$$

the last is due to (E) 4). This is the desired result. □

Now we are ready to state the main result of this section.

Theorem 4. *Under conditions* (E) *approximations* $\widetilde{X}$ *defined by* (43) *converge to the solution* X *of* (39) *in the mean-square sense, and moreover*

$$\mathsf{E}\left[(X(t)-\widetilde{X}(t))^2\right] \le C\delta^{2H}.$$

Proof. Estimate first for $h \in L^1(R)$

$$T_h Z(t) - Z(t) \le A_1 + A_2 + A_3$$

$$A_1 = \int_0^t T_h J_\sigma(s)\Big| b(s,(T_h J_\sigma^{-1}) T_h Z(s), \omega + h + M\sigma_s) - b(s,(T_h J_\sigma^{-1}) Z(s), \omega + h + M\sigma_s)\Big|\, ds,$$

$$A_2 = \int_0^t T_h J_\sigma(s)\Big| b(s,(T_h J_\sigma^{-1}) Z(s), \omega + h + M\sigma_s) - b(t,(T_h J_\sigma^{-1}(s)) Z(s), \omega + M\sigma_s)\Big|\, ds,$$

$$A_3 = \int_0^t \Big| T_h J_\sigma(s) b(t,(T_h J_\sigma^{-1}(s)) Z(s), \omega + M\sigma_s) - J_\sigma(s) b(t, J_\sigma^{-1}(s) Z(s), \omega + M\sigma_s)\Big|\, ds.$$

The condition (E) 1) gives $A_1 \le C\int_0^t |T_h Z(s) - Z(s)|\, ds$, the condition (E) 2) gives

$$A_2 \le C\int_0^T (1+|Z(s)|)\, ds \int_R |h(s)|\, ds$$

and Lemma 2 with boundedness of σ yields

$$\begin{aligned} A_3 &\le C\int_0^T (1 + J_\sigma(s) + T_h J(\sigma) + |Z(s)|)\, ds \left|\int_R M\sigma(s)\, h(s)\, ds\right|. \\ &\le C\int_0^T (1 + J_\sigma(s) + T_h J(\sigma) + |Z(s)|)\, ds \int_R |h(s)|\, ds. \end{aligned}$$

Applying Gronwall lemma, we get

$$|T_h Z(t) - Z(t)| \le C\int_0^T (1 + J_\sigma(s) + T_h J(\sigma) + |Z(s)|)\, ds \int_R |h(s)|\, ds.$$

Raising this inequality to the $2p$th power, taking expectations and using Jensen inequality and boundedness of moments of Z, J_σ and $T_h J_\sigma$ (the last follows from the Girsanov theorem, Cauchy–Schwartz inequality and assumptions on h), we get

$$\mathsf{E}\left[\left(T_h Z(t) - Z(t)\right)^{2p}\right] \le C(p)\left(\int_0^T |h(s)|\, ds\right)^{2p}.$$

Further,

$$\begin{aligned} &\mathsf{E}\left[\left(X(t)-\widetilde{X}(t)\right)^2\right] \le 3(A_1 + A_2 + A_3), \\ &A_1 = \mathsf{E}\left[\left(\overline{J}(t) T_{-M\widetilde{\sigma}\mathbf{1}_{[0,t]}}\big(Z(t) - \widetilde{Z}(t)\big)\right)^2\right], \\ &A_2 = \mathsf{E}\left[\left(\big(J_{-\sigma}(t) - \overline{J}(t)\big) T_{-M\widetilde{\sigma}\mathbf{1}_{[0,t]}} Z(t)\right)^2\right], \\ &A_3 = \mathsf{E}\left[\left(J_{-\sigma}(t)\big(T_{-M\sigma}(1 - T_{-M(\widetilde{\sigma}\mathbf{1}_{[0,t]} - \sigma_t)}\big) Z(t)\right)^2\right], \end{aligned}$$

where

$$J_{-\sigma}(t) = \exp\left\{\int_R M\sigma_t(s)dB^0_s - \frac{1}{2}\|\sigma_t\|^2_{\mathcal{H}}\right\},$$
$$\overline{J}(t) = \exp\left\{\int_R M(\widetilde{\sigma}\mathbf{1}_{[0,t]})(s)dB^0_s - \frac{1}{2}\left\|\widetilde{\sigma}\mathbf{1}_{[0,t]}\right\|^2_{\mathcal{H}}\right\}.$$

Now estimate using Cauchy–Schwartz inequality, Girsanov theorem (which can be applied as σ and $\widetilde{\sigma}$ are bounded on $[0,T]$) and Theorem 3

$$\begin{aligned} A_1 &\le \Big(\mathsf{E}\left[\overline{J}^4(t)\right]\mathsf{E}\left[T_{-M\widetilde{\sigma}\mathbf{1}_{[0,t]}}\big(Z(t)-\widetilde{Z}(t)\big)^4\right]\Big)^{1/2}, \\ &\le C\Big(\mathsf{E}\left[\widetilde{J}(t)\big(Z(t)-\widetilde{Z}(t)\big)^4\right]\Big)^{1/2} \\ &\le C\Big(\mathsf{E}\left[\widetilde{J}^2(t)\right]\mathsf{E}\left[\big(Z(t)-\widetilde{Z}(t)\big)^8\right]\Big)^{1/4} \le C\delta^{2H}. \end{aligned}$$

Similar reasoning and Lemma 3 imply

$$A_2 \le C\mathsf{E}\left[\left(\int_R M(\widetilde{\sigma}\mathbf{1}_{[0,t]}-\sigma_t)(s)\,dB^0_s + \frac{1}{2}\big(\|\sigma_t\|^2_{\mathcal{H}} - \left\|\widetilde{\sigma}\mathbf{1}_{[0,t]}\right\|^2_{\mathcal{H}}\big)\right)^2\right].$$

Using condition (E) 4), we obtain $A_2 \le C\delta^{2H}$. And for A_3, using the above estimate, we get

$$A_3 \le \int_0^t \left|M(\widetilde{\sigma}\mathbf{1}_{[0,t]}-\sigma_t)(s)\right| ds \le C\delta^{2H}.$$

This concludes the proof. □

Remark 4. It is natural to assume that the coefficient b is expressed in the terms of fBm B rather then in the terms of underlying Brownian motion B^0 (or underlying "Brownian" white noise ω.) This justifies the fact that it is σ not $M\sigma$ what is discretized in (41).

Remark 5. Similarly to the proof of Theorem 4 one can prove that for any $s \ge 1$

$$\mathsf{E}\left[\left|X(t)-\widetilde{X}(t)\right|^s\right] \le \delta^{sH}.$$

The case $s = 2$ is considered in the paper to keep classical "scent" of results.

Remark 6. Results of this section can be generalized for random initial condition X_0 in the following form: under conditions (E) and L^p-integrability of the initial condition one has convergence in any L^s for $s < p$ with

$$\mathsf{E}\left[\left|X(t)-\widetilde{X}(t)\right|^s\right] \le \delta^{sH}.$$

Proofs need some simple changes: Hölder inequality for appropriate powers instead of Cauchy–Schwartz one.

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